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Point and Interval Estimators, Based on m Order Statistics, for the Scale Parameter of a Weibull Population with Known Shape Parameter

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A derivation is given of the maximum likelihood estimator $\hat{\theta}$, based on the first m out of n ordered observations, of the scale parameter θ of a Weibull population with known shape parameter K. It is shown that $2m\,\hat{\theta}^K/\theta^K$ has a chi-square distribution with $2m$ degrees of freedom (independent of n). Use is made of this fact to set upper confidence bounds with confidence level $1 - P$ (lower confidence bounds with confidence level P) on the scale parameter θ. Formulas are given for the mean squared deviations of the upper and lower confidence bounds from the true value of the parameter. From these one can obtain expressions for the efficiency of confidence bounds and confidence intervals. The expected value of $\hat{\theta}$ is also determined, and from it the unbiasing factor $\tilde{\theta}/\hat{\theta}$ by which $\hat{\theta}$ must be multiplied to obtain an unbiased estimator $\tilde{\theta}$. An expression for the variance of the unbiased estimator $\tilde{\theta}$ is found. Values of the unbiasing factor and of the variance of the unbiased estimator, both of which are independent of n, are tabled for $m = 1(1)100$ and $K = 0.5(0.5)4.0(1.0)8.0$. A section on use of the table and a numerical example are included.

1. INTRODUCTION

Epstein and Sobel (1953) have pointed out the advantages of the use of ordered data from truncated tests to estimate the parameters of parent populations, and have worked out details for the exponential distribution. In particular, they have derived an estimator $\hat{\sigma}$ (which is maximum likelihood, unbiased, and minimum variance), based on the first m out of n ordered observations, of the parameter σ of an exponential population and have shown that $2m\hat{\sigma}/\sigma$ has a chi-square distribution with $2m$ degrees of freedom (independent of n). They have also given without derivation the maximum likelihood estimator $\hat{\theta}^K$, based on the first m out of n ordered observations, of θ^K, where θ is the scale parameter of a Weibull population with known shape parameter K. N. R. Mann (1963, p. 39) has stated without proof that $2m\hat{\theta}^K/\theta^K$ has a chi-square distribution with $2m$ degrees of freedom. The missing derivation and proof are supplied in the present paper. Expressions are given for upper and lower confidence bounds, $\bar{\theta}$ and $\underline{\theta}$, and for the efficiencies, as defined by Harter (1964b, c), of $\bar{\theta}$ and the central confidence interval $(\underline{\theta}, \bar{\theta})$. Brief discussions of the method of computation of the table and of its use are given, as well as a numerical example which illustrates the computation of both point and interval estimates and the efficiencies of both point and interval estimators.

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2. Maximum Likelihood Estimator for the Scale Parameter

The probability density function of the random variable Y having a Weibull distribution with location parameter 0, scale parameter θ, and shape parameter K is given by

$$(1) \qquad f(y) = \begin{cases} (K/\theta)(y/\theta)^{K-1} \exp\left[-(y/\theta)^K\right], & y > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Now if we define the random variable X by $X = Y^K$ and make the change of variable $x = y^K$ in (1), we find the probability density function of the random variable X to be

$$(2) \qquad g(x) = \begin{cases} \exp(-x/\theta^K)/\theta^K, & x > 0 \\ 0 & \text{elsewhere,} \end{cases}$$

which is the familiar exponential density function with parameter $\sigma = \theta^K$. Therefore if Y has a Weibull distribution with scale parameter θ and shape parameter K and if $X = Y^K$, then X is exponentially distributed with parameter $\sigma = \theta^K$. Hence a maximum likelihood m-order-statistic estimator for θ can be obtained from the "best" m-order-statistic estimator for $\sigma = \theta^k$ derived by Epstein and Sobel (1953), which is given by

$$(3) \qquad \hat{\sigma}_{mn} = [x_{1n} + x_{2n} + \cdots + x_{mn} + (n - m)x_{mn}]/m,$$

where x_{in} $(i = 1, 2, \cdots, m)$ are the first m order statistics of a sample of size n from an exponential population. Now, taking the K-th root of both sides of (3), we obtain

$$(4) \qquad \hat{\sigma}_{mn}^{1/K} = \{[x_{1n} + x_{2n} + \cdots + x_{mn} + (n - m)x_{mn}]/m\}^{1/K}.$$

Since $x_{in} = y_{in}^K$ we can write

$$(5) \qquad \hat{\theta}_{mn} = \hat{\sigma}_{mn}^{1/K} = \{[y_{1n}^K + y_{2n}^K + \cdots + y_{mn}^K + (n - m)y_{mn}^K]/m\}^{1/K}.$$

Now, since $\hat{\sigma}_{mn}$ is a maximum likelihood estimator of σ, $\hat{\theta}_{mn} = \hat{\sigma}_{mn}^{1/K}$ is a maximum likelihood estimator of $\theta = \sigma^{1/K}$.

The probability density function of the random variable $X_1 = \hat{\sigma}_{mn}$ is given by Epstein and Sobel (1953) as

$$(6) \qquad f_m(x_1) = \begin{cases} [1/\Gamma(m)](m/\sigma)^m x_1^{m-1} \exp(-mx_1/\sigma), & x_1 > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Now we let $\hat{\theta}_{mn} = \hat{\sigma}_{mn}^{1/K}$ or $Y_1 = X_1^{1/K}$ and we find the probability density function of $Y_1 = \hat{\theta}_{mn}$ to be

$$(7) \qquad g_m(y_1) = \begin{cases} [K/\Gamma(m)](m/\sigma)^m y_1^{Km-1} \exp(-my_1^K/\sigma), & y_1 > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Now, making the substitution $\sigma = \theta^K$ in (7), we obtain

$$g_m(y_1) = \begin{cases} [K/\Gamma(m)](m/\theta^K)^m y_1^{Km-1} \exp(-my_1^K/\theta^K), & y_1 > 0 \\ 0 \quad \text{elsewhere}. \end{cases} \tag{8}$$

Hereafter, for simplicity, we will denote $\hat{\theta}_{mn}$ by $\hat{\theta}$.

3. Confidence Bounds for Scale Parameter θ

From (8) it can be easily seen that $2m\hat{\theta}^K/\theta^K$ has a chi-square distribution with $2m$ degrees of freedom:

$$2m\hat{\theta}^K/\theta^K = \chi^2_{2m} . \tag{9}$$

Solving for θ, we obtain

$$\theta = (2m/\chi^2_{2m})^{1/K}\hat{\theta} \tag{10}$$

Then an upper confidence bound with confidence level $1 - P$ (lower confidence bound with confidence level P) on θ is given by

$$\bar{\theta}_{1-P} = \underline{\theta}_P = (2m/\chi^2_{2m,P})^{1/K}\hat{\theta}, \tag{11}$$

where the first subscript on χ^2 is the number of degrees of freedom and the second one is the cumulative probability. The interval between lower and upper confidence bounds, each with confidence level $1 - P$, will be call a central confidence interval with confidence level $1 - 2P$. Equations (9)-(11) remain valid when $m = n$, in which case (11) is an expression for the conventional confidence bound based on all n observations.

4. Efficiency of Confidence Bounds and Intervals

Harter (1964b, c) has defined the efficiency of a substitute upper confidence bound as the ratio, expressed as a percentage, of the mean squared deviation of the conventional upper confidence bound from the true parameter value to the mean squared deviation of the substitute upper confidence bound from the true parameter value. This definition may be expressed symbolically in the form

$$E_u = 100E[(\bar{\theta} - \theta)^2]/E[(\bar{\theta}' - \theta)^2], \tag{12}$$

where E_u is the efficiency (in percent) of the substitute upper confidence bound, $E[(\bar{\theta} - \theta)^2]$ is the mean squared deviation of the conventional upper confidence bound $\bar{\theta}$ from the true value θ of the parameter, and $E[(\bar{\theta}' - \theta)^2]$ is the mean squared deviation of the substitute upper confidence bound $\bar{\theta}'$ from the true value θ of the parameter. Further, the efficiency of a substitute central confidence interval is defined as the ratio, expressed as a percentage, of the sum of the mean squared deviations of the conventional upper and lower confidence bounds from the true parameter value to the sum of the mean squared deviations of the substitute upper and lower confidence bounds from the true parameter value. This definition in symbolic form is given by

$$E_c = 100\{E[(\bar{\theta} - \theta)^2] + E[(\underline{\theta} - \theta)^2]\}/\{E[(\bar{\theta}' - \theta)^2] + E[(\underline{\theta}' - \theta)^2]\}, \tag{13}$$

where E_c is the efficiency (in percent) of the substitute central confidence interval, $E[(\bar{\theta} - \theta)^2]$ and $E[(\bar{\theta}' - \theta)^2]$ are defined as above, and $E[(\underline{\theta} - \theta)^2]$

and $E[(\underline{\theta}' - \theta)^2]$ are respectively the mean squared deviations of the conventional and substitute lower confidence bounds $\underline{\theta}$ and $\underline{\theta}'$ from the true value θ of the parameter.

Quayle (1963) has shown that the mean squared deviation $E[(\bar{\theta} - \theta)^2]$ of the conventional upper confidence bound with confidence level $1 - P$, based on all n observations, from the true value θ of the scale parameter of a Weibull population with known shape parameter K is given by

$$(14) \qquad E[(\bar{\theta} - \theta)^2] = 1 - 2^{1+1/K}[\Gamma(n + 1/K)/\Gamma(n)][1/\chi^2_{2n,P}]^{1/K} + 2^{2/K}[\Gamma(n + 2/K)/\Gamma(n)] \times [1/\chi^2_{2n,P}]^{2/K}$$

and that the mean squared deviation $E[(\underline{\theta} - \theta)^2]$ of the corresponding conventional lower confidence bound is found by replacing $\bar{\theta}$ by $\underline{\theta}$ and P by $1 - P$ in (14). Since $2n\hat{\theta}_{nn}^K/\theta^K$ is distributed as χ^2 with $2n$ degrees of freedom and $2m\hat{\theta}^K/\theta^K$ as χ^2 with $2m$ degrees of freedom, the mean squared deviations of the substitute confidence bounds based on the first m order statistics are found as follows: $E[(\bar{\theta}' - \theta)^2]$ by replacing $\bar{\theta}$ by $\bar{\theta}'$ and n by m in (14); $E([\underline{\theta}' - \theta)^2]$ by replacing $\bar{\theta}$ by $\underline{\theta}'$, n by m, and P by $1 - P$ in (14). Substitution of the results in (12) and (13) then yields the efficiencies of the substitute upper confidence bound and the substitute central confidence interval, respectively, as compared with the conventional bound and interval based on all n observations.

5. Unbiased Estimator for the Scale Parameter

The expected value of $\hat{\theta}$ is found by using (8) to be

$$(15) \qquad E(\hat{\theta}) = \{\theta\Gamma(m + 1/K\}\{m^{1/K}\Gamma(m)\}.$$

Hence an unbiased estimator of θ is given by

$$(16) \qquad \tilde{\theta} = [m^{1/K}\Gamma(m)/\Gamma(m + 1/K)]\hat{\theta}.$$

The variance of the unbiased estimator $\tilde{\theta}$ is found to be

$$(17) \qquad \mathrm{Var}\ \tilde{\theta} = \theta^2\{[\Gamma(m)\Gamma(m + 2/K)/\Gamma^2(m + 1/K)] - 1\}.$$

Values of the unbiasing factor $\tilde{\theta}/\hat{\theta}$ for a maximum likelihood estimator and of the ratio $\mathrm{Var}\ \tilde{\theta}/\theta^2$ of the variance of an unbiased estimator to θ^2, expressions for which can be obtained by dividing both sides or (16) by $\hat{\theta}$ and both sides of (17) by θ^2, were computed for $m = 1(1)100$ and $K = 0.5(0.5)4.0(1.0)8.0$. The computations were performed on the IBM 1620 computer with FORTRAN programming, use being made of Stirling's approximation to the Gamma function. Twelve decimal digits were carried in the computations, but the values of the unbiasing factor were rounded to 6 decimal places (6 or 7 significant digits) and those of the variance to 8 decimal places (5 to 9 significant digits). The results are shown in Table 1.

6. Use of Table

In life-testing situations, one may wish to terminate the test without waiting for all n of the items placed on test to fail. If the life distribution is Weibull with known shape parameter K, where K is one of the values included in Table 1,

TABLE 1

Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population
Shape Parameter $K = 0.5$

m	$\hat{\theta}/\hat{\theta}$	Var $\hat{\theta}/\theta^2$	m	$\hat{\theta}/\hat{\theta}$	Var $\hat{\theta}/\theta^2$
1	.500000	5.0000000	51	.980769	.07918552
2	.666667	2.3333333	52	.981132	.07764877
3	.750000	1.5000000	53	.981481	.07617051
4	.800000	1.1000000	54	.981818	.07474747
5	.833333	.86666667	55	.982143	.07337662
6	.857143	.71428571	56	.982456	.07205514
7	.875000	.60714286	57	.982759	.07078040
8	.888889	.52777778	58	.983051	.06954997
9	.900000	.46666667	59	.983333	.06836158
10	.909091	.41818182	60	.983606	.06721311
11	.916667	.37878788	61	.983871	.06610259
12	.923077	.34615385	62	.984127	.06502816
13	.928571	.31868132	63	.984375	.06398810
14	.933333	.29523810	64	.984615	.06298077
15	.937500	.27500000	65	.984848	.06200466
16	.941176	.25735294	66	.985075	.06105834
17	.944444	.24183007	67	.985294	.06014047
18	.947368	.22807018	68	.985507	.05924979
19	.950000	.21578947	69	.985714	.05838509
20	.952381	.20476190	70	.985915	.05754527
21	.954545	.19480519	71	.986111	.05672926
22	.956522	.18577075	72	.986302	.05593607
23	.958333	.17753623	73	.986486	.05516475
24	.960000	.17000000	74	.986667	.05441441
25	.961538	.16307692	75	.986842	.05368421
26	.962963	.15669516	76	.987013	.05297334
27	.964286	.15079365	77	.987180	.05228105
28	.965517	.14532020	78	.987342	.05160662
29	.966667	.14022989	79	.987500	.05094937
30	.967742	.13548387	80	.987654	.05030864
31	.968750	.13104839	81	.987805	.04968383
32	.969697	.12689394	82	.987952	.04907435
33	.970588	.12299465	83	.988095	.04847963
34	.971429	.11932773	84	.988235	.04789916
35	.972222	.11587302	85	.988372	.04733242
36	.972973	.11261261	86	.988506	.04677894
37	.973684	.10953058	87	.988636	.04623824
38	.974359	.10661269	88	.988764	.04570991
39	.975000	.10384615	89	.988889	.04519351
40	.975610	.10121951	90	.989011	.04468864
41	.976190	.09872242	91	.989131	.04419494
42	.976744	.09634551	92	.989247	.04371201
43	.977273	.09408034	93	.989362	.04323953
44	.977778	.09191919	94	.989474	.04277716
45	.978261	.08985507	95	.989583	.04232456
46	.978723	.08788159	96	.989691	.04188144
47	.979167	.08599291	97	.989796	.04144751
48	.979592	.08418367	98	.989899	.04102247
49	.980000	.08244898	99	.990000	.04060606
50	.980392	.08078431	100	.990099	.04019802

TABLE 1 (*continued*)

Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population

Shape Parameter $K = 1.0$

m	$\hat{\theta}/\hat{\theta}$	Var $\hat{\theta}/\theta^2$	m	$\hat{\theta}/\hat{\theta}$	Var $\hat{\theta}/\theta^2$
1	1.000000	1.00000000	51	1.000000	.01960784
2	1.000000	.50000000	52	1.000000	.01923077
3	1.000000	.33333333	53	1.000000	.01886792
4	1.000000	.25000000	54	1.000000	.01851852
5	1.000000	.20000000	55	1.000000	.01818182
6	1.000000	.16666667	56	1.000000	.01785714
7	1.000000	.14285714	57	1.000000	.01754386
8	1.000000	.12500000	58	1.000000	.01724138
9	1.000000	.11111111	59	1.000000	.01694915
10	1.000000	.10000000	60	1.000000	.01666667
11	1.000000	.09090909	61	1.000000	.01639344
12	1.000000	.08333333	62	1.000000	.01612903
13	1.000000	.07692308	63	1.000000	.01587302
14	1.000000	.07142857	64	1.000000	.01562500
15	1.000000	.06666667	65	1.000000	.01538462
16	1.000000	.06250000	66	1.000000	.01515152
17	1.000000	.05882353	67	1.000000	.01492537
18	1.000000	.05555556	68	1.000000	.01470588
19	1.000000	.05263158	69	1.000000	.01449275
20	1.000000	.05000000	70	1.000000	.01428571
21	1.000000	.04761905	71	1.000000	.01408451
22	1.000000	.04545455	72	1.000000	.01388889
23	1.000000	.04347826	73	1.000000	.01369863
24	1.000000	.04166667	74	1.000000	.01351351
25	1.000000	.04000000	75	1.000000	.01333333
26	1.000000	.03846154	76	1.000000	.01315789
27	1.000000	.03703704	77	1.000000	.01298701
28	1.000000	.03571429	78	1.000000	.01282051
29	1.000000	.03448276	79	1.000000	.01265823
30	1.000000	.03333333	80	1.000000	.01250000
31	1.000000	.03225806	81	1.000000	.01234568
32	1.000000	.03125000	82	1.000000	.01219512
33	1.000000	.03030303	83	1.000000	.01204819
34	1.000000	.02941176	84	1.000000	.01190476
35	1.000000	.02857143	85	1.000000	.01176471
36	1.000000	.02777778	86	1.000000	.01162791
37	1.000000	.02702703	87	1.000000	.01149425
38	1.000000	.02631579	88	1.000000	.01136364
39	1.000000	.02564103	89	1.000000	.01123596
40	1.000000	.02500000	90	1.000000	.01111111
41	1.000000	.02439024	91	1.000000	.01098901
42	1.000000	.02380952	92	1.000000	.01086957
43	1.000000	.02325581	93	1.000000	.01075269
44	1.000000	.02272727	94	1.000000	.01063830
45	1.000000	.02222222	95	1.000000	.01052632
46	1.000000	.02173913	96	1.000000	.01041667
47	1.000000	.02127660	97	1.000000	.01030928
48	1.000000	.02083333	98	1.000000	.01020408
49	1.000000	.02040816	99	1.000000	.01010101
50	1.000000	.02000000	100	1.000000	.01000000

TABLE 1 (*continued*)

Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population

Shape Parameter $K = 1.5$

m	$\bar{\theta}/\hat{\theta}$	Var $\bar{\theta}/\theta^2$	m	$\bar{\theta}/\hat{\theta}$	Var $\bar{\theta}/\theta^2$
1	1.107732	.40099849	51	1.002179	.00872401
2	1.055040	.22723873	52	1.002137	.00855607
3	1.036879	.15053631	53	1.002096	.00839447
4	1.027710	.11250205	54	1.002058	.00823886
5	1.022187	.08979793	55	1.002020	.00808801
6	1.018498	.07471422	56	1.001984	.00794432
7	1.015800	.06396708	57	1.001949	.00780482
8	1.013880	.05592196	58	1.001916	.00767012
9	1.012340	.04967390	59	1.00188[illegible]	.00754000
10	1.011107	.04468139	60	1.001852	.00741422
11	1.010098	.04060061	61	1.001821	.00729257
12	1.009257	.03720273	62	1.001792	.00717484
13	1.008545	.03432959	63	1.001764	.00706085
14	1.007935	.03186847	64	1.001736	.00695043
15	1.007406	.02973941	65	1.001709	.00684341
16	1.006943	.02787170	66	1.001683	.00673964
17	1.006535	.02622710	67	1.001658	.00663897
18	1.006172	.02476585	68	1.001634	.00654126
19	1.005847	.02345875	69	1.001610	.00644648
20	1.005555	.02228270	70	1.001587	.00635422
21	1.005291	.02121803	71	1.001565	.00626465
22	1.005050	.02025209	72	1.001543	.00617758
23	1.004831	.01936952	73	1.001522	.00609289
24	1.004629	.01856066	74	1.001501	.00601040
25	1.004444	.01781684	75	1.001481	.00593020
26	1.004273	.01712987	76	1.001462	.00585221
27	1.004115	.01649427	77	1.001443	.00577615
28	1.003968	.01590405	78	1.001424	.00570204
29	1.003831	.01535462	79	1.001406	.00562981
30	1.003704	.01484188	80	1.001389	.00555939
31	1.003584	.01436227	81	1.001372	.00549071
32	1.003472	.01391270	82	1.001355	.00542371
33	1.003367	.01349011	83	1.001339	.00535832
34	1.003268	.01309300	84	1.001323	.00529449
35	1.003174	.01271833	85	1.001307	.00523216
36	1.003086	.01236452	86	1.001292	.00517128
37	1.003003	.01202985	87	1.001277	.00511180
38	1.002924	.01171282	88	1.001263	.00505368
39	1.002849	.01141208	89	1.001248	.00499686
40	1.002778	.01112639	90	1.001235	.00494131
41	1.002710	.01085465	91	1.001221	.00488697
42	1.002645	.01059587	92	1.001208	.00483382
43	1.002584	.01034914	93	1.001195	.00478181
44	1.002525	.01011365	94	1.001182	.00473091
45	1.002469	.00988863	95	1.001170	.00468109
46	1.002415	.00967340	96	1.001157	.00463230
47	1.002364	.00946735	97	1.001145	.00458451
48	1.002315	.00926989	98	1.001134	.00453771
49	1.002268	.00908049	99	1.001122	.00449185
50	1.002222	.00889869	100	1.001111	.00444690

TABLE 1 (*continued*)

Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population

Shape Parameter $K = 2.0$

m	$\bar{\theta}/\hat{\theta}$	Var $\bar{\theta}/\theta^2$	m	$\bar{\theta}/\hat{\theta}$	Var $\bar{\theta}/\theta^2$
1	1.128379	.27323954	51	1.002454	.00491392
2	1.063846	.13176848	52	1.002407	.00481919
3	1.042352	.08649774	53	1.002361	.00472805
4	1.031661	.06432432	54	1.002317	.00464030
5	1.025273	.05118452	55	1.002275	.00455574
6	1.021027	.04249704	56	1.002235	.00447421
7	1.018002	.03632842	57	1.002195	.00439554
8	1.015737	.03172251	58	1.002157	.00431959
9	1.013979	.02815254	59	1.002121	.00424623
10	1.012573	.02530447	60	1.002085	.00417531
11	1.011424	.02297952	61	1.002051	.00410672
12	1.010468	.02104572	62	1.002018	.00404035
13	1.009659	.01941205	63	1.001986	.00397610
14	1.008967	.01801368	64	1.001955	.00391385
15	1.008367	.01680320	65	1.001925	.00385352
16	1.007842	.01574513	66	1.001896	.00379503
17	1.007379	.01481240	67	1.001867	.00373828
18	1.006968	.01398398	68	1.001840	.00368320
19	1.006600	.01324330	69	1.001813	.00362973
20	1.006269	.01257713	70	1.001787	.00357778
21	1.005970	.01197477	71	1.001762	.00352730
22	1.005697	.01142746	72	1.001738	.00347823
23	1.005449	.01092799	73	1.001714	.00343050
24	1.005222	.01047035	74	1.001691	.00338407
25	1.005012	.01004949	75	1.001668	.00333887
26	1.004819	.00966116	76	1.001646	.00329487
27	1.004640	.00930172	77	1.001625	.00325201
28	1.004474	.00896807	78	1.001604	.00321025
29	1.004319	.00865752	79	1.001584	.00316955
30	1.004175	.00836776	80	1.001564	.00312987
31	1.004040	.00809677	81	1.001544	.00309117
32	1.003914	.00784278	82	1.001526	.00305341
33	1.003795	.00760423	83	1.001507	.00301657
34	1.003683	.00737977	84	1.001489	.00298061
35	1.003578	.00716818	85	1.001472	.00294549
36	1.003478	.00696839	86	1.001455	.00291119
37	1.003384	.00677943	87	1.001438	.00287768
38	1.003295	.00660045	88	1.001421	.00284493
39	1.003210	.00643067	89	1.001405	.00281292
40	1.003130	.00626941	90	1.001390	.00278163
41	1.003053	.00611604	91	1.001375	.00275102
42	1.002981	.00596999	92	1.001360	.00272107
43	1.002911	.00583076	93	1.001345	.00269178
44	1.002845	.00569787	94	1.001331	.00266310
45	1.002782	.00557090	95	1.001317	.10263503
46	1.002721	.00544947	96	1.001303	.00260755
47	1.002663	.00533322	97	1.001289	.00258063
48	1.002608	.00522183	98	1.001276	.00255427
49	1.002554	.00511499	99	1.001263	.00252843
50	1.002503	.00501244	100	1.001251	.00250312

TABLE 1 (*continued*)
Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population
Shape Parameter $K = 2.5$

m	$\bar{\theta}/\hat{\theta}$	Var $\bar{\theta}/\theta^2$	m	$\bar{\theta}/\hat{\theta}$	Var $\bar{\theta}/\theta^2$
1	1.127060	.18310455	51	1.002357	.00314830
2	1.062261	.08652458	52	1.002312	.00308754
3	1.041086	.05634335	53	1.002268	.00302909
4	1.030634	.04172268	54	1.002226	.00297281
5	1.024414	.03311340	55	1.002185	.00291859
6	1.020292	.02744474	56	1.002146	.00286630
7	1.017359	.02343128	57	1.002109	.00281586
8	1.015167	.02044098	58	1.002072	.00276716
9	1.013466	.01812705	59	1.002037	.00272012
10	1.012108	.01628345	60	1.002003	.00267465
11	1.010999	.01478008	61	1.001970	.00263067
12	1.010075	.01353073	62	1.001938	.00258812
13	1.009295	.01247607	63	1.001908	.00254692
14	1.008627	.01157389	64	1.001878	.00250701
15	1.008049	.01079335	65	1.001849	.00246834
16	1.007543	.01011142	66	1.001821	.00243084
17	1.007097	.00951052	67	1.001794	.00239446
18	1.006701	.00897702	68	1.001767	.00235916
19	1.006346	.00850019	69	1.001741	.00232488
20	1.006027	.00807145	70	1.001717	.00229158
21	1.005739	.00768388	71	1.001692	.00225922
22	1.005477	.00733182	72	1.001669	.00222777
23	1.005238	.00701061	73	1.001646	.00219717
24	1.005019	.00671635	74	1.001624	.00216741
25	1.004818	.00644580	75	1.001602	.00213844
26	1.004632	.00619820	76	1.001581	.00211024
27	1.004460	.00596521	77	1.001560	.00208277
28	1.004300	.00575082	78	1.001540	.00205601
29	1.004151	.00555131	79	1.001521	.00202992
30	1.004012	.00536517	80	1.001502	.00200449
31	1.003882	.00519111	81	1.001483	.00197969
32	1.003761	.00502799	82	1.001465	.00195549
33	1.003647	.00487481	83	1.001447	.00193188
34	1.003539	.00473069	84	1.001430	.00190884
35	1.003438	.00459484	85	1.001413	.00188633
36	1.003342	.00446657	86	1.001397	.00186435
37	1.003251	.00434528	87	1.001381	.00184288
38	1.003166	.00423039	88	1.001365	.00182189
39	1.003084	.00412143	89	1.001350	.00180138
40	1.003007	.00401793	90	1.001335	.00178133
41	1.002933	.00391951	91	1.001320	.00176171
42	1.002863	.00382579	92	1.001306	.00174253
43	1.002797	.00373645	93	1.001292	.00172375
44	1.002733	.00365119	94	1.001278	.00170538
45	1.002672	.00356973	95	1.001264	.00168740
46	1.002614	.00349183	96	1.001251	.00166979
47	1.002558	.00341725	97	1.001238	.00165254
48	1.002505	.00334579	98	1.001226	.00163565
49	1.002454	.00327726	99	1.001213	.00161910
50	1.002404	.00321149	100	1.0012[illegible]1	.00160288

TABLE 1 (*continued*)

Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population

Shape Parameter $K = 3.0$

m	$\bar{\theta}/\hat{\theta}$	Var $\bar{\theta}/\theta^2$	m	$\bar{\theta}/\hat{\theta}$	Var $\bar{\theta}/\theta^2$
1	1.119847	.13209336	51	1.002183	.00218813
2	1.058189	.06133753	52	1.002141	.00214587
3	1.038277	.03967758	53	1.002101	.00210521
4	1.028495	.02928080	54	1.002062	.00206607
5	1.022689	.02319038	55	1.002024	.00202835
6	1.018846	.01919354	56	1.001988	.00199199
7	1.016115	.01637029	57	1.001953	.00195691
8	1.014075	.01427035	58	1.001919	.00192304
9	1.012494	.01264752	59	1.001887	.00189032
10	1.011231	.01135588	60	1.001855	.00185870
11	1.010201	.01030348	61	1.001825	.00182812
12	1.009343	.00942952	62	1.001795	.00179853
13	1.008619	.00869217	63	1.001767	.00176988
14	1.007998	.00806173	64	1.001739	.00174213
15	1.007461	.00751654	65	1.001712	.00171524
16	1.006992	.00704040	66	1.001686	.00168916
17	1.006578	.00662097	67	1.001661	.00166387
18	1.006210	.00624870	68	1.001637	.00163932
19	1.005882	.00591606	69	1.001613	.00161549
20	1.005586	.00561704	70	1.001590	.00159234
21	1.005319	.00534678	71	1.001567	.00156984
22	1.005076	.00510133	72	1.001546	.00154797
23	1.004854	.00487743	73	1.001524	.00152670
24	1.004651	.00467235	74	1.001504	.00150601
25	1.004464	.00448383	75	1.001484	.00148587
26	1.004292	.00430992	76	1.001464	.00146626
27	1.004132	.00414900	77	1.001445	.00144716
28	1.003984	.00399966	78	1.001427	.00142856
29	1.003846	.00386070	79	1.001408	.00141042
30	1.003717	.00373107	80	1.001391	.00139274
31	1.003597	.00360986	81	1.001374	.00137550
32	1.003484	.00349628	82	1.001357	.00135868
33	1.003378	.00338962	83	1.001340	.00134227
34	1.003279	.00328928	84	1.001324	.00132625
35	1.003185	.00319471	85	1.001309	.00131060
36	1.003096	.00310543	86	1.001294	.00129533
37	1.003012	.00302100	87	1.001279	.00128040
38	1.002932	.00294104	88	1.001264	.00126581
39	1.002857	.00286520	89	1.001250	.00125155
40	1.002785	.00279318	90	1.001236	.00123761
41	1.002717	.00272469	91	1.001222	.00122398
42	1.002652	.00265947	92	1.001209	.00121064
43	1.002591	.00259731	93	1.001196	.00119760
44	1.002532	.00253798	94	1.001183	.00118483
45	1.002475	.00248131	95	1.001171	.00117232
46	1.002421	.00242711	96	1.001159	.00116008
47	1.002370	.00237523	97	1.001147	.00114810
48	1.002320	.00232551	98	1.001135	.00113636
49	1.002273	.00227784	99	1.001124	.00112485
50	1.002227	.00223208	100	1.001112	.00111358

TABLE 1 (*continued*)

Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population

Shape Parameter $K = 3.5$

m	$\tilde{\theta}/\hat{\theta}$	Var $\tilde{\theta}/\theta^2$	m	$\tilde{\theta}/\hat{\theta}$	Var $\tilde{\theta}/\hat{\theta}^2$
1	1.111423	.10014607	51	1.002006	.00160864
2	1.053765	.04581787	52	1.001967	.00157756
3	1.035293	.02947697	53	1.001930	.00154765
4	1.026247	.02169264	54	1.001894	.00151885
5	1.020886	.01715179	55	1.001859	.00149111
6	1.017341	.01417983	56	1.001826	.00146436
7	1.014824	.01208442	57	1.001794	.00143856
8	1.012945	.01052796	58	1.001763	.00141365
9	1.011488	.00932639	59	1.001733	.00138958
10	1.010326	.00837081	60	1.001704	.00136633
11	1.009378	.00759275	61	1.001676	.00134383
12	1.008589	.00694696	62	1.001649	.00132207
13	1.007922	.00640237	63	1.001623	.00130100
14	1.007351	.00593692	64	1.001597	.00128059
15	1.006857	.00553455	65	1.001573	.00126081
16	1.006426	.00518324	66	1.001549	.00124164
17	1.006045	.00487386	67	1.001526	.00122304
18	1.005707	.00459932	68	1.001503	.00120498
19	1.005405	.00435406	69	1.001481	.00118745
20	1.005133	.00413362	70	1.001460	.00117043
21	1.004887	.00393443	71	1.001440	.00115389
22	1.004664	.00375355	72	1.001420	.00113780
23	1.004460	.00358857	73	1.001400	.00112216
24	1.004273	.00343748	74	1.001381	.00110695
25	1.004101	.00329859	75	1.001363	.00109214
26	1.003943	.00317049	76	1.001345	.00107772
27	1.003796	.00305197	77	1.001327	.00106368
28	1.003660	.00294199	78	1.001310	.00104999
29	1.003533	.00283966	79	1.001294	.00103666
30	1.003415	.00274421	80	1.001277	.00102366
31	1.003305	.00265496	81	1.001262	.00101098
32	1.003201	.00257134	82	1.001246	.00099862
33	1.003104	.00249283	83	1.001231	.00098655
34	1.003012	.00241896	84	1.001217	.00097477
35	1.002926	.00234935	85	1.001202	.00096327
36	1.002844	.00228363	86	1.001188	.00095203
37	1.002767	.00222149	87	1.001175	.00094106
38	1.002694	.00216264	88	1.001161	.00093033
39	1.002625	.00210683	89	1.001148	.00091985
40	1.002559	.00205382	90	1.001135	.00090960
41	1.002496	.00200342	91	1.001123	.00089958
42	1.002437	.00195543	92	1.001111	.00088977
43	1.002380	.00190969	93	1.001099	.00088018
44	1.002326	.00186604	94	1.001087	.00087079
45	1.002274	.00182434	95	1.001075	.00086160
46	1.002224	.00178446	96	1.001064	.00085260
47	1.002177	.00174629	97	1.001053	.00084379
48	1.002131	.00170971	98	1.001043	.00083515
49	1.002088	.00167464	99	1.001032	.00082670
50	1.002046	.00164098	100	1.001022	.00081841

TABLE 1 (*continued*)
Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population
Shape Parameter $K = 4.0$

m	$\tilde{\theta}/\hat{\theta}$	Var $\tilde{\theta}/\theta^2$	m	$\tilde{\theta}/\hat{\theta}$	Var $\tilde{\theta}/\theta^2$
1	1.103263	.07870520	51	1.001843	.00123225
2	1.049606	.03555699	52	1.001807	.00120843
3	1.032516	.02277234	53	1.001773	.00118551
4	1.024163	.01672043	54	1.001740	.00116344
5	1.019220	.01320237	55	1.001709	.00114218
6	1.015954	.01090486	56	1.001678	.00112168
7	1.013635	.00928741	57	1.001648	.00110190
8	1.011905	.00808731	58	1.001620	.00108281
9	1.010564	.00716161	59	1.001592	.00106437
10	1.009495	.00642591	60	1.001566	.00104655
11	1.008622	.00582721	61	1.001540	.00102932
12	1.007896	.00533050	62	1.001515	.00101264
13	1.007283	.00491179	63	1.001491	.00099649
14	1.006758	.00455404	64	1.001468	.00098086
15	1.006304	.00424485	65	1.001445	.00096570
16	1.005907	.00397497	66	1.001423	.00095101
17	1.005557	.00373734	67	1.001402	.00093675
18	1.005246	.00352652	68	1.001381	.00092292
19	1.004968	.00333820	69	1.001361	.00090949
20	1.004718	.00316898	70	1.001342	.00089645
21	1.004492	.00301608	71	1.001323	.00088377
22	1.004286	.00287725	72	1.001304	.00087145
23	1.004099	.00275064	73	1.001287	.00085946
24	1.003927	.00263470	74	1.001269	.00084781
25	1.003769	.00252814	75	1.001252	.00083646
26	1.003624	.00242987	76	1.001236	.00082541
27	1.003489	.00233894	77	1.001220	.00081465
28	1.003364	.00225458	78	1.001204	.00080417
29	1.003247	.00217609	79	1.001189	.00079396
30	1.003138	.00210288	80	1.001174	.00078400
31	1.003037	.00203443	81	1.001159	.00077428
32	1.002942	.00197030	82	1.001145	.00076481
33	1.002852	.00191009	83	1.001131	.00075556
34	1.002768	.00185345	84	1.001118	.00074654
35	1.002688	.00180007	85	1.001105	.00073773
36	1.002614	.00174968	86	1.001092	.00072912
37	1.002543	.00170204	87	1.001079	.00072071
38	1.002476	.00165692	88	1.001067	.00071250
39	1.002412	.00161413	89	1.001055	.00070447
40	1.002351	.00157349	90	1.001043	.00069862
41	1.002294	.00153485	91	1.001032	.00068894
42	1.002239	.00149806	92	1.001020	.00068143
43	1.002187	.00146300	93	1.001009	.00067408
44	1.002137	.00142954	94	1.000999	.00066688
45	1.002089	.00139757	95	1.000988	.00065984
46	1.002044	.00136701	96	1.000978	.00065295
47	1.002000	.00133775	97	1.000968	.00064620
48	1.001958	.00130972	98	1.000958	.00063959
49	1.001918	.00128283	99	1.000948	.00063311
50	1.001880	.00125703	100	1.000939	.00062676

TABLE 1 (*continued*)

Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population

Shape Parameter $K = 5.0$

m	$\tilde{\theta}/\hat{\theta}$	Var $\tilde{\theta}/\theta^2$	m	$\tilde{\theta}/\hat{\theta}$	Var $\tilde{\theta}/\theta^2$
1	1.089124	.05246525	51	1.001573	.00078924
2	1.042563	.02323010	52	1.001543	.00077397
3	1.027845	.01477365	53	1.001513	.00075928
4	1.020674	.01080969	54	1.001485	.00074514
5	1.016435	.00851761	55	1.001458	.00073151
6	1.013637	.00702572	56	1.001432	.00071837
7	1.011653	.00597782	57	1.001407	.00070570
8	1.010172	.00520161	58	1.001383	.00069347
9	1.009025	.00460363	59	1.001359	.00068165
10	1.008110	.00412886	60	1.001336	.00067023
11	1.007304	.00374281	61	1.001314	.00065918
12	1.006744	.00342274	62	1.001293	.00064850
13	1.006219	.00315307	63	1.001273	.00063815
14	1.005771	.00292278	64	1.001253	.00062813
15	1.005383	.002[illegible]383	65	1.001233	.00061842
16	1.005043	.002[illegible]023	66	1.001215	.00060900
17	1.004744	.00239742	67	1.001197	.00059087
18	1.004479	.00226189	68	1.001179	.00059101
19	1.004241	.00214086	69	1.001162	.00058240
20	1.004028	.00203212	70	1.001145	.00057404
21	1.003835	.00193389	71	1.001129	.00056592
22	1.003659	.00184472	72	1.001113	.00055803
23	1.003499	.00176341	73	1.001098	.00055035
24	1.003353	.00168896	74	1.001083	.00054288
25	1.003218	.00162054	75	1.001069	.00053561
26	1.003093	.00155745	76	1.001055	.00052853
27	1.002978	.00149909	77	1.001041	.00052164
28	1.002871	.00144494	78	1.001027	.00051493
29	1.002772	.00139457	79	1.001014	.00050838
30	1.002679	.00134759	80	1.001002	.00050200
31	1.002592	.00130367	81	1.000989	.00049578
32	1.002511	.00126253	82	1.000977	.00048971
33	1.002434	.00122390	83	1.000965	.00048379
34	1.002363	.00118757	84	1.000954	.00047801
35	1.002295	.00115333	85	1.000943	.00047236
36	1.002231	.00112101	86	1.000932	.00046685
37	1.002170	.00109045	87	1.000921	.00046146
38	1.002113	.00106151	88	1.000911	.00045620
39	1.002059	.00103407	89	1.000900	.00045106
40	1.002007	.00100802	90	1.000890	.00044603
41	1.001958	.00098324	91	1.000880	.00044111
42	1.001911	.00095965	92	1.000871	.00043630
43	1.001867	.00093717	93	1.000862	.00043159
44	1.001824	.00091571	94	1.000852	.00042698
45	1.001783	.00089522	95	1.000843	.00042247
46	1.001744	.00087562	96	1.000835	.00041806
47	1.001707	.00085687	97	1.000826	.00041373
48	1.001672	.00083890	98	1.000817	.00040950
49	1.001637	.00082167	99	1.000809	.00040535
50	1.001604	.00080513	100	1.000801	.00040128

TABLE 1 (*continued*)

Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population

Shape Parameter $K = 6.0$

m	$\tilde{\theta}/\hat{\theta}$	Var $\tilde{\theta}/\theta^2$	m	$\tilde{\theta}/\hat{\theta}$	Var $\tilde{\theta}/\theta^2$
1	1.077912	.03754820	51	1.001366	.00054838
2	1.037070	.01637374	52	1.001339	.00053776
3	1.024223	.01035970	53	1.001314	.00052755
4	1.017974	.00756092	54	1.001289	.00051772
5	1.014284	.00594882	55	1.001266	.00050825
6	1.011850	.00490205	56	1.001243	.00049911
7	1.010124	.00416801	57	1.001221	.00049030
8	1.008837	.00362492	58	1.001200	.00048180
9	1.007840	.00320692	59	1.001180	.00047359
10	1.007045	.00287528	60	1.001160	.00046565
11	1.006396	.00260576	61	1.001141	.00045797
12	1.005857	.00238241	62	1.001123	.00045054
13	1.005401	.00219431	63	1.001105	.00044335
14	1.005012	.00203373	64	1.001088	.00043639
15	1.004674	.00189504	65	1.001071	.00042964
16	1.004380	.00177405	66	1.001055	.00042309
17	1.004120	.00166758	67	1.001039	.00041675
18	1.003889	.00157317	68	1.001023	.00041059
19	1.003683	.00148887	69	1.001009	.00040461
20	1.003497	.00141314	70	1.000994	.00039880
21	1.003330	.00134474	71	1.000980	.00039315
22	1.003177	.00128266	72	1.000966	.00038767
23	1.003038	.00122605	73	1.000953	.00038233
24	1.002914	.00117423	74	1.000940	.00037714
25	1.002794	.00112661	75	1.000928	.00037209
26	1.002686	.00108271	76	1.000915	.00036717
27	1.002586	.00104269	77	1.000904	.00036238
28	1.002493	.00100442	78	1.000892	.00035771
29	1.002407	.00096937	79	1.000881	.00035317
30	1.002326	.00093668	80	1.000870	.00034873
31	1.002251	.00090613	81	1.000859	.00034441
32	1.002180	.00087751	82	1.000848	.00034019
33	1.002114	.00085064	83	1.000838	.00033607
34	1.002051	.00082537	84	1.000828	.00033206
35	1.001992	.00080155	85	1.000818	.00032813
36	1.001937	.00077907	86	1.000809	.00032430
37	1.001884	.00075782	87	1.000800	.00032056
38	1.001834	.00073769	88	1.000790	.00031690
39	1.001787	.00071861	89	1.000782	.00031333
40	1.001742	.00070049	90	1.000773	.00030983
41	1.001700	.00068326	91	1.000764	.00030642
42	1.001659	.00066686	92	1.000756	.00030307
43	1.001620	.00065123	93	1.000748	.00029980
44	1.001584	.00063631	94	1.000740	.00029660
45	1.001548	.00062206	95	1.000732	.00029347
46	1.001514	.00060843	96	1.000724	.00029040
47	1.001482	.00059539	97	1.000717	.00028740
48	1.001451	.00058290	98	1.000710	.00028445
49	1.001421	.00057092	99	1.000702	.00028157
50	1.001393	.00055942	100	1.000695	.00027874

TABLE 1 (*continued*)

Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population

Shape Parameter $K = 7.0$

m	$\bar{\theta}/\hat{\theta}$	Var $\bar{\theta}/\theta^2$	m	$\bar{\theta}/\hat{\theta}$	Var $\bar{\theta}/\theta^2$
1	1.069018	.02823145	51	1.001204	.00040305
2	1.032756	.01216533	52	1.001181	.00039525
3	1.021387	.00766682	53	1.001158	.00038774
4	1.015863	.00558486	54	1.001137	.00038051
5	1.012604	.00438916	55	1.001116	.00037354
6	1.010454	.00361417	56	1.001096	.00036685
7	1.008931	.00307138	57	1.001077	.00036035
8	1.007794	.00267015	58	1.001058	.00035410
9	1.006915	.00236154	59	1.001040	.00034806
10	1.006213	.00211683	60	1.001023	.00034222
11	1.005641	.00191803	61	1.001006	.00033658
12	1.005165	.00175335	62	1.000990	.00033112
13	1.004763	.00161470	63	1.000974	.00032583
14	1.004420	.00149636	64	1.000959	.00032071
15	1.004122	.00139418	65	1.000944	.00031575
16	1.003862	.00130506	66	1.000930	.00031094
17	1.003633	.00122664	67	1.000916	.00030627
18	1.003430	.00115711	68	1.000902	.00030174
19	1.003248	.00109504	69	1.000889	.00029735
20	1.003084	.00103929	70	1.000877	.00029308
21	1.002936	.00098893	71	1.000864	.00028893
22	1.002802	.00094324	72	1.000852	.00028490
23	1.002679	.00090157	73	1.000840	.00028097
24	1.002567	.00086343	74	1.000829	.00027716
25	1.002464	.00082839	75	1.000818	.00027344
26	1.002368	.00079608	76	1.000807	.00026983
27	1.002280	.00076620	77	1.000797	.00026631
28	1.002198	.00073848	78	1.000786	.00026288
29	1.002122	.00071269	79	1.000776	.00025953
30	1.002051	.00068864	80	1.000767	.00025628
31	1.001984	.00066617	81	1.000757	.00025310
32	1.001922	.00064511	82	1.000748	.00025000
33	1.001864	.00062534	83	1.000739	.00024697
34	1.001809	.00060675	84	1.000730	.00024402
35	1.001757	.00058924	85	1.000722	.00024114
36	1.001708	.00057270	86	1.000713	.00023832
37	1.001661	.00055707	87	1.000705	.00023557
38	1.001618	.00054227	88	1.000697	.00023288
39	1.001576	.00052823	89	1.000689	.00023025
40	1.001536	.00051491	90	1.000681	.00022768
41	1.001499	.00050224	91	1.000674	.00022517
42	1.001463	.00049017	92	1.000667	.00022272
43	1.001429	.00047868	93	1.000659	.00022031
44	1.001396	.00046771	94	1.000652	.00021796
45	1.001365	.00045723	95	1.000645	.00021565
46	1.001335	.00044721	96	1.000639	.00021340
47	1.001307	.00043762	97	1.000632	.00021119
48	1.001279	.00042843	98	1.000626	.00020903
49	1.001253	.00041962	99	1.000619	.00020691
50	1.001228	.00041117	100	1.000613	.00020483

TABLE 1 (*continued*)
Unbiasing Factors for Maximum Likelihood Estimators and Variances of Unbiased Estimators from m Order Statistics of Scale Parameter θ of Weibull Population
Shape Parameter $K = 8.0$

m	$\tilde{\theta}/\hat{\theta}$	Var $\tilde{\theta}/\theta^2$	m	$\tilde{\theta}/\hat{\theta}$	Var $\tilde{\theta}/\theta^2$
1	1.061801	.02201333	51	1.001076	.00030868
2	1.029305	.00930588	52	1.001055	.00030270
3	1.019123	.00599316	53	1.001035	.00029695
4	1.014180	.00429371	54	1.001016	.00029141
5	1.011265	.00337150	55	1.000997	.00028607
6	1.009343	.00277461	56	1.000979	.00028093
7	1.007980	.00235606	57	1.000962	.00027597
8	1.006965	.00204845	58	1.000945	.00027118
9	1.006178	.00181128	59	1.000929	.00026656
10	1.005551	.00162328	60	1.000914	.00026208
11	1.005040	.00147062	61	1.000899	.00025776
12	1.004615	.00134419	62	1.000884	.00025358
13	1.004256	.00123770	63	1.000870	.00024953
14	1.003949	.00114695	64	1.000857	.00024560
15	1.003683	.00106854	65	1.000843	.00024180
16	1.003450	.00100017	66	1.000831	.00023812
17	1.003246	.00094002	67	1.000818	.00023454
18	1.003064	.00088669	68	1.000806	.00023108
19	1.002901	.00083908	69	1.000794	.00022771
20	1.002755	.00079633	70	1.000783	.00022444
21	1.002623	.00075772	71	1.000772	.00022126
22	1.002503	.00072268	72	1.000761	.00021817
23	1.002393	.00069074	73	1.000751	.00021517
24	1.002293	.00066150	74	1.000741	.00021224
25	1.002201	.00063463	75	1.000731	.00020940
26	1.002116	.00060987	76	1.000721	.00020663
27	1.002037	.00058696	77	1.000712	.00020393
28	1.001964	.00056571	78	1.000702	.00020131
29	1.001896	.00054595	79	1.000694	.00019875
30	1.001832	.00052752	80	1.000685	.00019625
31	1.001773	.00051029	81	1.000676	.00019381
32	1.001717	.00049415	82	1.000668	.00019144
33	1.001665	.00047901	83	1.000660	.00018912
34	1.001616	.00046476	84	1.000652	.00018686
35	1.001569	.00045133	85	1.000645	.00018465
36	1.001526	.00043866	86	1.000637	.00018250
37	1.001484	.00042669	87	1.000630	.00018039
38	1.001445	.00041534	88	1.000623	.00017833
39	1.001408	.00040459	89	1.000616	.00017632
40	1.001372	.00039438	90	1.000609	.00017435
41	1.001339	.00038467	91	1.000602	.00017243
42	1.001307	.00037543	92	1.000595	.00017054
43	1.001276	.00036662	93	1.000589	.00016870
44	1.001247	.00035822	94	1.000583	.00016690
45	1.001219	.00035019	95	1.000577	.00016514
46	1.001193	.00034251	96	1.000571	.00016341
47	1.001167	.00033516	97	1.000565	.00016172
48	1.001143	.00032813	98	1.000559	.00016006
49	1.001120	.00032138	99	1.000553	.00015844
50	1.001097	.00031490	100	1.000548	.00015685

and if the test is terminated at the time of the m-th failure ($m \leq 100$), one can compute a maximum likelihood estimator $\hat{\theta}$ of the scale parameter θ from (5) and then multiply $\hat{\theta}$ by the unbiasing factor $\bar{\theta}/\hat{\theta}$ given in Table 1 to obtain an unbiased estimator $\bar{\theta}$. The ratio, Var $\bar{\theta}/\theta^2$, of the variance of the unbiased estimator to θ^2 is also given in the table. The efficiency E_p of the unbiased estimator based on the first m order statistics as compared with the one based on all n order statistics ($m<n\leq 100$) can be found by taking the ratio of two entries in the Var $\bar{\theta}/\theta^2$ columns of Table 1. It can be seen that the percentage efficiency is approximately $100m/n$.

7. Numerical Example

As an illustration of the use of the above results, consider a simulated life test on forty components. Suppose the observed failure times in hours are as follows:

5	33	55	65	82	102	114	142
10	34	58	65	85	103	116	143
17	36	58	66	90	106	117	151
32	54	61	67	92	107	124	158
32	55	64	68	92	114	139	195

Suppose the experimenter knows that these times are from a Weibull population with shape parameter $K = 2.0$ and wishes to obtain a point estimate and set 80% upper and lower confidence bounds on the scale parameter θ. The conventional confidence bounds are those based on all 40 observations, but the experimenter might not want to wait for all the components to fail and might therefore terminate the test at the time of the mth failure ($m < 40$). We can simulate this occurrence by censoring upper portions of the above ordered data. The values of the maximum likelihood estimator $\hat{\theta}$ were calculated from (5) for $m = 8(8)40$, and $\bar{\theta}$ was obtained by multiplying by the unbiasing factor $\bar{\theta}/\hat{\theta}$ given in Table 1. Then the lower and upper 80% confidence bounds, $\underline{\theta}_{.80}$ and $\bar{\theta}_{.80}$, were calculated from (11) with the aid of a table of percentage points of the chi-square distribution given by Harter (1964a). The intervals between paired values of $\underline{\theta}_{.80}$ and $\bar{\theta}_{.80}$ are central 60% confidence intervals for θ. The efficiencies, E_u and E_i, of upper confidence bounds and central confidence intervals, with confidence levels 80% and 60%, respectively, based on the first m out of n ordered observations, were calculated by substituting, in (12) and (13), values of $E[(\bar{\theta}' - \theta)^2]$ obtained from (11) and of $E[(\bar{\theta}' - \theta)^2]$, $E[(\underline{\theta} - \theta)^2]$, and $E[(\underline{\theta}' - \theta)^2]$ obtained from (11) modified as indicated in Section 4. The efficiency E_p of the unbiased point estimator $\bar{\theta}$ was computed as indicated in Section 7. The results are as follows:

m	$\hat{\theta}$	$\bar{\theta}$	$\underline{\theta}_{.80}$	$\bar{\theta}_{.80}$	$E_u(\%)$	$E_i(\%)$	$E_p(\%)$
8	77.0	78.2	68.1	92.2	16.2	18.2	19.8
16	91.9	92.6	83.5	103.3	34.4	38.9	39.8
24	95.2	95.7	88.2	104.8	57.3	59.3	59.9
32	93.7	94.1	87.6	101.7	78.5	79.6	74.9
40	93.3	93.6	87.8	100.3	100.0	100.0	100.0

Note that $E_u \leq E_s \leq E_p \leq 100m/n$ and that $E_u \to E_s \to E_p \to 100m/n \to 100\%$ as $m \to n$.

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13. ABSTRACT

A derivation is given of the maximum likelihood estimator $\hat{\theta}$, based on the first m out of n ordered observations, of the scale parameter θ of a Weibull population with known shape parameter K. It is shown that $2m\hat{\theta}^K/\theta^K$ has a chi-square distribution with 2m degrees of freedom (independent of n). Use is made of this fact to set upper confidence bounds with confidence level 1−P (lower confidence bounds with confidence level P) on the scale parameter θ. Formulas are given for the mean squared deviations of the upper and lower confidence bounds from the true value of the parameter. From these one can obtain expressions for the efficiency of confidence bounds and confidence intervals. The expected value of $\hat{\theta}$ is also determined, and from it the unbiasing factor $\tilde{\theta}/\hat{\theta}$ by which $\hat{\theta}$ must be multiplied to obtain an unbiased estimator $\tilde{\theta}$. An expression for the variance of the unbiased estimator $\tilde{\theta}$ is found. Values of the unbiasing factor and of the variance of the unbiased estimator, both of which are independent of n, are tabled for m = 1(1)100 and K = 0.5(0.5)4.0(1.0)8.0. A section on use of the table and a numerical example are included.

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